Sains Malaysiana 43(8)(2014): 1271–1274

Solving Linear Multi-Objective Geometric Programming Problems via Reference Point Approach

(Menyelesaikan Masalah Linear Berbilang Objektif Geometri Pengaturcaraan melalui Pendekatan Titik Rujukan)

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ABSTRACT

In the last few years we have seen a very rapid development on solving generalized geometric programming (GGP) problems, but so far less works has been devoted to MOGP due to the inherent difficulty which may arise in solving such problems. Our aim in this paper was to consider the problem of multi-objective geometric programming (MOGP) and solve the problem via two-level relaxed linear programming problem Yuelin et al. (2005) and that is due to simplicity which occurs through linearization i.e. transforming a GP to LP. In this approach each of the objective functions in multi-objective geometric programming is individually linearized using two-level linear relaxed bound method, which provides a lower bound for the optimal values. Finally our MOGP is transformed to a multi-objective linear programming problem (MOLP) which is solved by reference point approach. In the end, a numerical example is given to investigate the feasibility and effectiveness of the proposed approach.

Keywords: Geometric programming; linearization technique; multi-objective programming; reference point method

ABSTRAK

Sejak beberapa tahun lepas, kita telah melihat pembangunan yang sangat pesat dalam masalah penyelesaian am geometri pengaturcaraan (GGP), tetapi setakat terdapat kurang kajian tentang MOGP kerana wujud kesukaran yang mungkin timbul dalam menyelesaikan masalah tersebut. Matlamat kami dalam kertas ini adalah untuk mempertimbangkan masalah pengaturcaraan pelbagai objektif geometri (MOGP) dan menyelesaikan masalah ini melalui masalah pengaturcaraan linear santai dua peringkat Yuelin et al. (2005) dan yang adalah kerana kemudahan yang berlaku melalui pelinearan iaitu transformasi GP ke LP. Dalam pendekatan ini, setiap satu daripada fungsi objektif dalam pengaturcaraan pelbagai objektif geometri secara individu adalah dilinearkan menggunakan kaedah terikat santai linear dua peringkat, yang memberikan sesuatu had lebih rendah bagi nilai yang optimum. Akhirnya MOGP ini berubah menjadi sebuah masalah pengaturcaraan linear pelbagai objektif (MOLP) yang diselesaikan melalui pendekatan titik rujukan. Akhirnya contoh berangka diberikan untuk mengkaji kemungkinan dan keberkesanan pendekatan yang dicadangkan.

Kata kunci: Kaedah titik rujukan; pelbagai objektif pengaturcaraan; pengaturcaraan geometri; pelinearan teknik

INTRODUCTION

The general form of multi-objective geometric programming problem is defined as follows:

$$\begin{cases} \min & G_0(x) \\ \vdots & \vdots \\ \min & G_k(x) \\ s.t & \\ & G_i(x) \le \delta_i, \quad i = k+1,\dots m \\ & x \in \Omega \end{cases}$$

$$\Omega \equiv \{x_j : 0 \le \underline{x}_j \le x_j \le \overline{x}_j < \infty, 1 \le j \le n\}$$

$$\underline{x}_j < \overline{x}_j, \ \forall j \in N G_i(x) = \sum_{t=1}^{T_i} \alpha_{it} \prod_{j=1}^{J_{it}} x_j^{\gamma_{itj}}, \ \forall i = 0, 1, \dots, m.$$

The T_i is stand for the index set of terms in $G_i(x)$ and each term $t \in T_i$ has a nonzero real coefficient α_{it} and a combination products of monomials $x_j^{\gamma_{itj}}$ for $j \in J_{it}$ which is subset of $N = \{1, \dots, n\}$. There are distinct indices used in J_{it} and the γ_{itj} is supposed to be nonzero, and $\underline{x}_j > 0$ if $\gamma_{itj} < 0$ and the $|J_{it}|$ represents the number of elements in J_{it}. As we know multi-objective geometric programming problems consist of several objective functions, in which each of the objective is in the form of geometric programming. The geometric programming problems are usually considered in class of nonlinear programming problems. So far we have seen many different algorithms for solving and experimenting geometric programming problems and many good works is introduced by researchers in different field of sciences such as mechanic engineering and chemical engineering. More over in the recent years we have seen many useful results in solving geometric programming problems by different investigators. The problem of profit maximization is considered by Li and Chen (2010). Quantity discount is involved in profit maximization is the work of Liu (2006). Multi-objective marketing planning inventory model is discussed by Sahidul (2008). The global optimization for signomial geometric programming was investigated by Shen et al. (2008). Wu (2008) discussed the problem of optimizing the geometric programming with single-term exponent subject to max-min fuzzy relational equation constraints. A good work on linearization is done by Qu et al. (2008) in which they considered the problem of a global optimization using linear relaxation for generalized geometric programming. Global optimization of signomial geometric programming via linear relaxation is considered by Shen and Zhang (2004).

Linear Relaxation Technique

The main structure in the development of a solution procedure for solving problem (MOGP) is the construction of lower bounds for this problem. A linear programming relaxation problem can be solved to obtain a lower bound to the solution of problem. The proposed strategy for generating this linear programming relaxation is to underestimate every nonlinear function $G_i(x)(i = 0, 1, \ldots, m)$. Let $\Omega_j \equiv \{x_j : \underline{x}_j \le x_j \le \overline{x}_j\}$, we use $L_{itj}^{\Omega_j}(x_j), U_{itj}^{\Omega_j}(x_j)$ to denote the nonnegative lower and upper bound of linear approximate functions $x_j^{\gamma_{itj}}$ over Ω_j for $j \in J_{it}$. $l_{itj}^{\Omega_j}(x_j)$ is to denote the straight line through points $(\underline{x}_j, \underline{x}_j^{\gamma_{itj}})$ and $(\overline{x}_j, \overline{x}_j^{\gamma_{itj}})$, $l_{itj}^{*\Omega_j}(x_j)$ to denote the straight line that is tangent to $x_j^{\gamma_{itj}}$ at $(x_{itj}^*, x_{itj}^{*\gamma_{itj}})$ where:

$$x_{itj}^* = \left[\frac{\gamma_{itj}(\overline{x}_j - \underline{x}_j)}{\overline{x}_{itj}^{\gamma_{itj}} - \underline{x}_{itj}^{\gamma_{itj}}}\right]^{\frac{1}{(1-\gamma_{itj})}}, \quad \text{as} \quad \gamma_{itj} \ne 1.$$

$$l_{itj}^{\Omega_j}(x_j) = \underline{x}_j^{\gamma_{itj}} + \frac{\overline{x}_j^{\gamma_{itj}} - \underline{x}_j^{\gamma_{itj}}}{\overline{x}_j - \underline{x}_j}(x_j - \underline{x}_j)$$

$$l_{itj}^{*\Omega_j}(x_j) = x_{itj}^{*\gamma_{itj}} + \frac{\overline{x}_j^{\gamma_{itj}} - \underline{x}_j^{\gamma_{itj}}}{\overline{x}_j - \underline{x}_j}(x_j - x_{itj}^*)$$

We need to introduce the following two straight lines $\tilde{l}_{itj}^{\Omega_j}(x_j)$ and $\hat{l}_{itj}^{\Omega_j}(x_j)$, where $\tilde{l}_{itj}^{\Omega_j}(x_j)$ passes through the point $(\underline{x}_j, 0)$ and is tangent to $x_j^{\gamma_{itj}}$ at the point $(\tilde{x}_{itj}, \tilde{x}_{itj}^{\gamma_{itj}})$ and $\hat{l}_{itj}^{\Omega_j}(x_j)$ passes through the point $(\overline{x}_j, 0)$ and is tangent to $x_j^{\gamma_{itj}}$ at the point $(\hat{x}_{itj}, \hat{x}_{itj}^{\gamma_{itj}})$ where:

$$\tilde{x}_{itj} = \left(\frac{\gamma_{itj}}{\gamma_{itj} - 1}\right)\underline{x}_j, \ (\text{as } \gamma_{itj} > 1)$$

$$\tilde{l}_{itj}^{\Omega_j}(x_j) = \gamma_{itj}\tilde{x}_{itj}^{\gamma_{itj}-1}(x_j - \underline{x}_j), \quad \text{as} \quad \gamma_{itj} > 1$$

and

$$\hat{x}_{itj} = \left(\frac{\gamma_{itj}}{\gamma_{itj} - 1}\right)\overline{x}_j, \ (\text{as } \gamma_{itj} < 0)$$

$$\hat{l}_{itj}^{\Omega_j}(x_j) = \gamma_{itj}\hat{x}_{itj}^{\gamma_{itj}-1}(x_j - \overline{x}_j), \quad \text{as} \quad \gamma_{itj} < 0.$$

First-Stage Relaxation

If $\alpha_{it} > 0$, then

$$L_{itj}^{\Omega_j}(x_j) = \begin{cases} l_{itj}^{*\Omega_j}(x_j) & when \ \ \gamma_{itj} > 1 \ \ \& \ \ l_{itj}^{*\Omega_j}(\underline{x}_j) \ge 0 \\ \tilde{l}_{itj}^{\Omega_j}(x_j) & when \ \ \gamma_{itj} > 1 \ \ \& \ \ l_{itj}^{*\Omega_j}(\underline{x}_j) < 0 \\ x_j & when \ \ \gamma_{itj} = 1 \\ l_{itj}^{\Omega_j}(x_j) & when \ \ 0 < \gamma_{itj} < 1 \\ l_{itj}^{*\Omega_j}(x_j) & when \ \ \gamma_{itj} < 0 \ \ \& \ \ l_{itj}^{*\Omega_j}(\overline{x}_j) \ge 0 \\ \hat{l}_{itj}^{\Omega_j}(x_j) & when \ \ \gamma_{itj} < 0 \ \ \& \ \ l_{itj}^{*\Omega_j}(\overline{x}_j) < 0 \end{cases}.$$

If $\alpha_{it} < 0$, then

$$U_{itj}^{\Omega_j}(x_j) = \begin{cases} l_{itj}^{\Omega_j}(x_j) & when \ \ \gamma_{itj} > 1 \quad or \quad \gamma_{itj} < 0 \\ l_{itj}^{*\Omega_j}(x_j) & when \ \ 0 < \gamma_{itj} < 1 \\ x_j & when \ \ \gamma_{itj} = 1 \end{cases}.$$

Therefore we can get a lower bounded function for the $\alpha_{it}\prod_{j=1}^{J_{it}} x_j^{\gamma_{itj}}$.

$$\alpha_{it}\prod_{j=1}^{J_{it}} x_j^{\gamma_{itj}} \ge G_{it}^{R(\Omega)}(x).$$

When

$$G_i^{R(\Omega)}(x) \equiv \begin{cases} \alpha_{it}\prod_{j=1}^{J_{it}} L_{itj}^{\Omega_j}(x_j) & if \quad \alpha_{it} > 0 \\ \alpha_{it}\prod_{j=1}^{J_{it}} U_{itj}^{\Omega_j}(x_j) & if \quad \alpha_{it} < 0 \end{cases} \quad \forall i, t.$$

We have over all the terms $t \in T_i$ for each i = 0, 1, ..., m, $\sum_{i=1}^{T_i} G_{it}^{R(\Omega)}(x)$ as $G_i^{R(\Omega)}(x)$ and $G_i(x) \ge G_i^{R(\Omega)}(x)$, $\forall x \in \Omega$, i=0, 1,....,m . Therefore

$$FLR(\Omega): \begin{cases} \min & G_0^{R(\Omega)}(x) \\ \vdots & \vdots \\ \min & G_k^{R(\Omega)}(x) \\ s.t & G_i^{R(\Omega)}(x) \le \delta_i \qquad i = k+1, \ldots, m \\ & x \in \Omega \end{cases}.$$

Second-Stage Relaxation

Theorem1. The function $l(y) = \prod_{j=1}^{p} y_j$ has lower and upper bounded linear functions $q_{11}(y)$, $q_{12}(y)$ and $q_{21}(y)$, $q_{22}(y)$ over $\overline{R} = \{y \in R^p : \underline{\beta}_j \le y_j \le \overline{\beta}_j, j = 1, 2, \ldots, p\}$.

$$\begin{cases} q_{11}(y)=\sum_{j=1}^{p}\left(\prod_{\substack{i=1\\ i\neq j}}^{p}\underline{\beta}_i\right)y_j-(p-1)\prod_{j=1}^{p}\underline{\beta}_j \\ q_{12}(y)=\sum_{j=1}^{p}\left(\prod_{\substack{i=1\\ i\neq j}}^{p}\overline{\beta}_i\right)y_j-(p-1)\prod_{j=1}^{p}\overline{\beta}_j \\ q_{21}(y)=\sum_{j=1}^{p}\left(\prod_{i=1}^{j-1}\overline{\beta}_i\right)\left(\prod_{i=j+1}^{p}\underline{\beta}_i\right)y_j-\sum_{j=1}^{p-1}\left(\prod_{i=1}^{j}\overline{\beta}_i\right)\left(\prod_{j=1}^{p}\underline{\beta}_i\right) \\ q_{22}(y)=\sum_{j=1}^{p}\left(\prod_{i=1}^{j-1}\underline{\beta}_i\right)\left(\prod_{i=j+1}^{p}\overline{\beta}_i\right)y_j-\sum_{j=1}^{p-1}\left(\prod_{i=1}^{j}\underline{\beta}_i\right)\left(\prod_{i=j+1}^{p}\overline{\beta}_i\right) \end{cases},$$

and $l(y) = q_{11}(y)$ for all $y\in\{\underline{\beta}\}\cup N_1$ where N_1 denotes the set of all extreme points of $\overline{R}$ adjacent to $\underline{\beta}$, $l(y) = q_{12}(y)$ for all where N_2 denotes the set of all extreme points of $\overline{R}$ adjacent to $\overline{\beta}$ (note that in the above expression, we have used the notations $\prod_{i=1}^{p} w_i = 1$ and $\prod_{i=i+1}^{p} w_i = 1$).

For proof, Yuelin et al. 2005.

Now, if $L_{itj}^{\Omega_j}(x_j)$ is increasing over $[\underline{x}_j,\overline{x}_j]$, then let $\underline{L}_{itj}=L_{itj}^{\Omega_j}(\underline{x}_j)$, otherwise $\overline{L}_{itj}=L_{itj}^{\Omega_j}(\overline{x}_j)$, let $\underline{L}_{itj}=L_{itj}^{\Omega_j}(\overline{x}_j)$, $\overline{L}_{itj}=L_{itj}^{\Omega_j}(\underline{x}_j)$. If $U_{itj}^{\Omega_j}(x_j)$ is increasing over $[\underline{x}_j,\overline{x}_j]$, then let $\underline{U}_{itj}=U_{itj}^{\Omega_j}(\underline{x}_j)$, $\overline{U}_{itj}=U_{itj}^{\Omega_j}(\overline{x}_j)$, otherwise, let $\underline{U}_{itj}=U_{itj}^{\Omega_j}(\overline{x}_j)$, $\overline{U}_{itj}=U_{itj}^{\Omega_j}(\underline{x}_j)$.

Then

$$G_{it}^{R(\Omega)}(x) \geq \overline{G}_{it}^{R(\Omega)}(x),\ \ \forall x\in\Omega,\ \ i=0,1,\ldots,m.$$

When $|J_{it}|>1$,

$$\overline{G}_{it}^{R(\Omega)}(x)=\begin{cases} \alpha_{it}\left(\sum_{j=1}^{J_{it}}\left(\prod_{\substack{k=1\\ k\neq j}}^{J_{it}}\underline{L}_{itk}\right)L_{itj}^{\Omega_j}(x_j)-(|J_{it}|-1)\prod_{k=1}^{J_{it}}\underline{L}_{itk}\right) & if\ \ \alpha_{it}>0 \\ \alpha_{it}\left(\sum_{j=1}^{J_{it}}\left(\prod_{\substack{k=1\\ k<j}}^{J_{it}}\overline{U}_{itk}\right)\left(\prod_{\substack{k=1\\ k>j}}^{J_{it}}\underline{U}_{itk}\right)U_{itj}^{\Omega_j}(x_j)-\right. & \\ \left.\sum_{j=1}^{J_{it}\{\max J_{it}\}}\left(\prod_{\substack{k=1\\ k\leq j}}^{J_{it}}\overline{U}_{itk}\right)\left(\prod_{\substack{k=1\\ k>j}}^{J_{it}}\underline{U}_{itk}\right)\right) & if\ \ \alpha_{it}<0 \end{cases}$$

and when $|J_{it}|=1$,

$$\overline{G}_{it}^{R(\Omega)}(x)=\begin{cases} L_{itj}^{\Omega_j}(x_j) & if\ \ \alpha_{it}>0 \\ U_{itj}^{\Omega_j}(x_j) & if\ \ \alpha_{it}<0 \end{cases}\ \ \forall i,t.$$

and

$$\tilde{G}_{it}^{R(\Omega)}(x)=\sum_{i=1}^{T_i}G_{it}^{R(\Omega)}(x),\ \ \forall t\in T_i,\ j\in J_{it},\ i=0.1\ldots,m.$$

Thus the linear relaxation programming problem (MOGP) over Ω can be described as follows:

$$SLR(\Omega):\begin{cases} \min & \tilde{G}_0^{R(\Omega)}(x) \\ \vdots & \vdots \\ \min & \tilde{G}_0^{R(\Omega)}(x) \\ s.t. & \tilde{G}_0^{R(\Omega)}(x)\leq\delta_i \qquad i=k+1,\ldots,m \\ & x\in\Omega \end{cases}.$$

For demonstrating the behavior of the optimal objective function value, the following relation is always true.

V[SLR(Ω)] ≤ V[FLR(Ω)] ≤ V[MOGP(Ω)].

Where V[MOGP(Ω)] stands for the primal objective function value, V[FLR(Ω)] the value of objective function in the first-stage of linear relaxation, whereas V[SLR(Ω)] represents the value of objective function in the second-stage of the linear relaxation problem. As we have mentioned in the title of the present article, we use one of the practical interactive method called the reference point method (RPM). In this approach the DM (decision maker) plays an important role rather than primary role in obtaining the desired value of the objective function. In this procedure, when the DM specifies a reference point, the corresponding scalarization problem is solved for generating the Pareto optimal solution which is, in a sense, close to the reference point or better than that, if the reference point is attainable. Then the DM either chooses the current Pareto optimal solution or modifies the reference point to find a satisfying solution. Let us assume that the DM is unable to specify a proper reference point due to complexity of the problem. Then the corresponding Pareto optimal solution, which is in the minimax sense, nearest to the reference point or better than that if the reference point is attainable, is obtained by solving the following minimax problem:

$$\begin{cases} \min & \max\{z_i(x)-\overline{z}_i\} \\ & i=1,\ldots k \\ s.t & \\ & x\in X \end{cases},$$

or equivalently

$$\begin{cases} \min & \lambda \\ s.t & \\ & z_i(x)-z_i\leq\lambda \\ x\in X, & i=1,\ldots,k \end{cases}.$$

Numerical Experiment

Consider the following example,

$$\begin{cases} \min & z_1(x) = 25x_1^2 + 30x_1^{-2}x_2 \\ \min & z_2(x) = 15x_1 + 20x_1^{-1}x_2^2 \\ s.t & \\ & x_1^{-1}x_2^{-1} \le 1 \\ & (1,1) \le (x_1, x_2) \le (2,2) \end{cases}.$$

Then by using linear relaxation technique we obtain the first-stage approximating to the problem as:

$$FLR(\Omega): \begin{cases} \min & z_1(x) = 75x_1 - 22.5x_1x_2 + 121.9 \\ \min & z_2(x) = 37.5x_1 - 30x_1x_2 \\ s.t & \\ & -0.21x_1 - 0.21x_2 \le 3\times 10^{-9} \\ & (1,1) \le (x_1, x_2) \le (2,2) \end{cases},$$

and second -stage approximation problem is given as follows:

$$SLR(\Omega): \begin{cases} \min & z_1(x) = -19.5x_1 + 1.80x_2 + 51.75 \\ \min & z_2(x) = 7.5x_1 + 24.85x_2 + 2.26 \\ s.t & \\ & -0.21x_1 - 0.21x_2 \le 3\times 10^{-9} \\ & (1,1) \le (x_1, x_2) \le (2,2) \end{cases}.$$

Now each of the objective function is individually solve for max and min to get the reference point idea for the DM.

$$\begin{cases} z_1^{\min} = 14.55 \\ x_1 = 2 \\ x_2 = 1 \end{cases}, \quad \begin{cases} z_1^{\max} = 35.85 \\ x_1 = 1 \\ x_2 = 2 \end{cases}$$

$$\begin{cases} z_2^{\min} = 34.61 \\ x_1 = 1 \\ x_2 = 1 \end{cases}, \quad \begin{cases} z_2^{\max} = 66.96 \\ x_1 = 2 \\ x_2 = 2 \end{cases}$$

$\bar{z}_1 = 14.55, \quad \bar{z}_2 = 34.61$

$$\begin{cases} \min & \lambda \\ s.t & \\ & -19.5x_1 + 1.80x_2 - \lambda \le -37.2 \\ & 7.5x_1 + 24.85x_2 - \lambda \le 32.35 \\ & -0.21x_1 - 0.21x_2 \le 3\times 10^{-9} \\ & (1,1) \le (x_1, x_2) \le (2,2) \end{cases}$$

$$\begin{cases} \lambda = 5.416667 \\ x_1^* = 1.722222 \\ x_2^* = 1 \end{cases}$$

$$\begin{cases} z_1^* = z_1(x_1^*, x_2^*) = 19.966671 \\ z_2^* = z_2(x_1^*, x_2^*) = 40.026665 \end{cases}$$

$$\begin{cases} z_1^* - \bar{z}_1 = 5.416667 \\ z_2^* - \bar{z}_2 = 5.416667 \end{cases}$$

CONCLUSION

In this paper a linear relaxation method is used to transform a multi-objective nonlinear programming problem to a linear programming problem. This linearization is done in two stages and by using the reference point method in the second stage, an approximate lower bound solution is obtained which is very closed to the optimum value of the original problem.

REFERENCES

Li, Y.M. & Chen, Y-C. 2010. Temperature -aware floor planning via geometric programming. *Mathematical and Computer Modeling* 51: 927-934.

Liu, S-T. 2006. A geometric programming approach to profit maximization. *Applied Mathematics and Computation* 182: 1093-1097.

Sahidul, I. 2008. Multi-objective marketing planning inventory model: A geometric programming approach. *Applied Mathematics and Computation* 205: 238-246.

Shen, P-P., Li, X-A. & Jiao, H-W. 2008. Accelerating method of global optimization for signomial geometric programming. *Journal of Computational and Applied Mathematics* 214: 66-77.

Shen, P. & Zhang, K. 2004. Global optimization of signomial geometric programming using linear relaxation. *Applied Mathematics and Computation* 150: 99-114.

Qu, S., Zhang, K. & Wang, F. 2008. A global optimization using linear relaxation for generalized geometric programming. *European Journal of Operational Research* 190: 345-356.

Wu, Y-K. 2008. Optimizing the geometric programming problem with single-term exponents subject to max-min fuzzy relational equation constraints. *Mathematical and Computer Modelling* 47: 352-362.

Yuelin, G., Chengxian, X., Yanjun, W. & Lianshen, Z. 2005. A new two-level linear relaxed bound method for geometric programming problems. *Applied Mathematics and Computation* 164: 117-131.

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Received: 15 June 2013
Accepted: 16 December 2013